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A

THEORY OF THE NEGATIVE SIGN;

IN WHICH, FROM

PRINCIPLES GENERAL AND NEW IN APPLICATION,

THE ALGEBRA

OF

POSITIVE AND NEGATIVE QUANTITIES

IS

SIMPLY AND LOGICALLY DEDUCED.

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PREFACE.

THE usual method of teaching algebra is first to lay down definitions, purely arithmetical, of the symbols and operations to be employed; symbolical rules founded on these definitions are then deduced and exhibited in their most general form, but the exceptions to their arithmetical application, arising from the limited character of the principles involved, are either concealed or insufficiently explained: thus, in processes having a show of demonstration, the student is led unconsciously to violate the rules of reasoning, and when with notions purely arithmetical he looks for intelligible results, or seeks to verify results obtained, *negative quantities* appear, and he learns, for the first time perhaps, that addition does not necessarily imply increase, that subtraction does not necessarily imply diminution, and that other quantities besides numbers are to be treated by arithmetical rules. Some rough idea of the meaning of isolated negative quantities is then given, and perhaps particular examples to teach the general mode of interpreting negative results, and he is told, and required by experiment to convince himself, that the unconscious use of negative quantities in processes supposed and intended to be arithmetical, does not vitiate the results. He is thus led to regard algebra as a science in which, by some imperfectly understood properties of the symbolical rules, contradictions produce consistencies, and errors in reasoning produce true results; and thus conviction arising from logical deduction is replaced by faith. The object of this work is to lay down definitions at once simple and general, which will not be contradicted by processes afterwards employed, and so place algebra, as far as positive and negative quantities are concerned, on the same logical footing as geometry, or any other science in which no idea or process is admissible which the first principles do not include. There can be no doubt that the objections stated are in part the cause that geometry has been so much exalted in preference to algebra as a discipline for the mind, the general idea being that algebra wants that strictness of deduction which in geometry is supplied.

One great hindrance to adequate and clear conceptions of the principles of Algebra is the want of proper distinction in most elementary works between values abstract and concrete. It is not sufficiently pointed out that the algebras of the two are not identical; that the sign + has a meaning in $A+B$ when A and B are concrete, distinct and different from that which it has in $m+n$ when m and n are abstract: and while all combinations of m, n, &c., with the signs +, −, ×, ÷, are possible, those combinations only of A, B, &c. are intelligible which are made with + and −. The general plan I have pursued is, first, accurately to distinguish between concrete quantities and their ratios to one another; secondly, to assign their proper meanings to $A+B$ and $m\times A$ for all *abstract* values of m, and for all *concrete* values of A and B; thirdly, to define $a+b$, $a-b$, $a\times b$, and $a\div b$ for all *abstract* values of a and b in the manner following:—

I. The ratio of $aU+bU$ to U shown to be constant for all real *concrete* values of U, is denoted by $a+b$; so that $aU+bU = (a+b)U$.

II. The ratio of $a(bU)$ to U shown to be constant for all real *concrete* values of U, is denoted by $a\times b$; so that $a(bU) = (a\times b)U$.

III. The signs − and ÷ indicate operations whose effects are destroyed by the operations + and ×; so that $a-b+b = a$ and $a\div b\times b = a$.

Thus the equations $aU+bU = (a+b)U$ and $a(bU) = (a\times b)U$ are *symbolical definitions* of the

[illegible]

[illegible]

[illegible]

distinction of *positive* and *negative* to *concrete* magnitudes would do well to consider what notion consistently with this restriction they would attach to such expressions as $(-5)\times(-4)$, bearing in mind that though we may give a meaning to $m\times A$ when A only is concrete, no meaning whatever when both A and B are concrete can in general be assigned to $A\times B$. If it be said that the relations of positive and negative magnitudes to one another with respect both to quantity and affection should not be *called* ratios, it then becomes a question of *names* only, while the *principles* contended for remain.

In Art. 35, page 31, I have supplied what, for some unaccountable reason and often much to the perplexity of the student, is usually omitted in elementary works, viz., a concise and general definition of the terms *greater* and *less*, as applied to values positive and negative. How simply and briefly the fundamental relations of inequality may be deduced from this definition will be seen in the same article.

The proper order of study for those who would acquire a satisfactory and competent knowledge of theoretical algebra, is, first, Arithmetical Algebra, in which positive values only are admitted; secondly, signs $+$ and $\times$ when used with abstract values a and b, or as they may be called the ***rules of interpretation*** of those signs; and not, as they are usually considered, ***propositions*** deducible from previously defined meanings of $a+b$ and $a\times b$.

From these principles all the fundamental relations between positive and negative values are deduced, and the identity of this algebra with general arithmetic so far as positive values only are concerned is manifest.

There are some who object to the distinction of *positive* and *negative* as applied to *ratios*, and in some works of acknowledged authority the notion is specially discarded, and this on the ground not merely that it is unadvisable or inconvenient, but that it is inadmissible, inasmuch as *ratio* is a relation of magnitudes to others *of the same kind*. But concrete magnitudes, positive and negative, though *oppositely affected*, are necessarily *of the same kind*, and as such have with respect to *quantity* irrespective of *affection* the relation one to another in which ratio consists; and the distinction of these ratios into positive and negative according to the affection of the quantities to which they are referred does not alter their nature, though it classifies them in a convenient way. Those who would confine the distinction of *positive* and *negative* to *concrete* magnitudes would do well to consider what notion consistently with this restriction they would attach to such expressions as $(-5) \times (-4)$, bearing in mind that though we may give a meaning to $m \times A$ when A only is concrete, no meaning whatever when both A and B are concrete can in general be assigned to $A \times B$. If it be said that the relations of positive and negative magnitudes to one another with respect both to quantity and affection should not be *called* ratios, it then becomes a question of *names* only, while the *principles* contended for remain.

In Art. 35, page 31, I have supplied what, for some unaccountable reason and often much to the perplexity of the student, is usually omitted in elementary works, viz., a concise and general definition of the terms *greater* and *less*, as applied to values positive and negative. How simply and briefly the fundamental relations of inequality may be deduced from this definition will be seen in the same article.

The proper order of study for those who would acquire a satisfactory and competent knowledge of theoretical algebra, is, first, Arithmetical Algebra, in which positive values only are admitted; secondly, Single Algebra, in which $-a$ is explained; thirdly, Double Algebra, in which $\sqrt{-a}$ and other symbols inadmissible in Single Algebra receive their due interpretation. Those learners who think this method of progressive teaching too tedious may pass over the first two, and *try* to satisfy themselves with an algebra in which no unexplained results are likely to arise, by at once proceeding to the third. The attempt would be about as hopeful as the study of Euclid's Elements for the first time by commencing with the sixth book, and beneficial much in the same way, viz., by affording a practical lesson on the powers of the human intellect, and a more assured conviction that proficiency in this as in other studies is to be arrived at only by degrees.

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THEORY OF THE NEGATIVE SIGN;

OR,

THE ALGEBRA

OF

POSITIVE AND NEGATIVE QUANTITIES.

ART. 1. The Concrete Quantities of this algebra are quantities oppositely affected, but of the same kind, such as—money gained and money lost; time past and time future; distance north and distance south; and such like.

Such quantities are relatively to each other distinguished as Positive and Negative. Thus, if money gained be called Positive, then money lost is Negative; but if money lost be called Positive, then money gained is Negative: the terms Positive and Negative not being expressive of any qualities inherent in the quantities themselves, but being terms of Relation, indicating opposition *of some sort* among the quantities concerned.

Quantities all positive or all negative are said to be Like; and quantities positive and quantities negative, as distinguished from one another, are said to be Unlike.

Concrete quantities positive or negative will be denoted by capitals A, B, C, &c.

2. The Ratio of a concrete quantity to a like quantity will be called Positive; and the Ratio of a concrete quantity to an unlike quantity will be called Negative.

There is, then, this distinction in the use of the terms Positive and Negative as applied to concrete quantities and their ratios; viz. — that applied to concrete quantities they are used relatively; a quantity being negative, not in itself, but relatively to other quantities which are called positive—but applied to ratios they are used absolutely, every ratio being, irrespective of other ratios, either positive or negative in itself.

3. The Ratios of concrete quantities positive and negative will be denoted by small letters, a, b, c, &c. But, in order more definitely to express the Values of such ratios, we shall for present purposes use the notation following, viz., numerals or accented letters, a', b', c', &c., to express the

numerical values of the ratios concerned, and subscript letters p and n to denote the class, positive or negative, to which they are referred—thus, a'_p is a positive ratio whose numerical value is a'; b'_n is a negative ratio whose numerical value is b'. The operations addition, subtraction, &c., and the relations equal, greater, and less, with their respective signs, are, when applied to the numerical symbols a', b', c', &c., to be arithmetically understood.

aU will be used to denote a quantity whose ratio to U is a. Hence, agreeably with the above notation, a'_nU is a quantity which has to U a negative ratio whose numerical value is a'; a'_nU is, therefore, a quantity unlike to U, and in relation of magnitude as $a' : 1$.

Since quantities which have the same numerical ratio to a given quantity are arithmetically equal, it is evident that aU has but one value for the same values of a and U.

4. Zero values with respect to concrete quantities and ratios will be denoted respectively by O and o, and will for convenience be classed as concrete quantities and ratios with those of real value.

When either a or U is zero, the quantity aU will be zero: and to avoid unnecessary exceptions, a zero ratio will be included with other ratios as the ratio of zero quantities to those of real value.

5. **Definition.** A = B denotes that A and B are either both zero quantities, or like quantities of the same magnitude: and $a = b$ denotes that a and b are either both zero ratios, or like ratios numerically equal.

Note. The sign = is to be read "equal to."

Cor. 1. Concrete quantities or ratios equal to the same concrete quantities or ratios are equal to one another.

Cor. 2. If $\mathrm{U} = \mathrm{V}$, then $a\mathrm{U} = a\mathrm{V}$; and if $a\mathrm{U} = b\mathrm{U}$ then $a = b$.

1 stands for the ratio of equality: so that $1\mathrm{U} = 1_p\mathrm{U} = \mathrm{U}$. (Art. 3).

6. Belonging to every concrete quantity or set of quantities zero, positive and negative, is a quantity we shall call their Nullifier, which is defined as follows.

Definition. When the quantities are all zero, their Nullifier is zero; and in other cases, it is that quantity zero, positive or negative, which equalizes the total magnitudes of the real quantities of each class.

Thus, the nullifier of the five quantities 5_nU, 6_pU, O, 3_nU, 12_pU is 10_nU; because the annexation of 10_nU makes the total magnitudes of the positive and negative quantities the same; that is, nullifies their difference or reduces it to nothing, whence the nullifier takes its name.

7. **Definition.** A+B is the quantity whose nullifier is that of the quantities A and B.

Cor. 1. When A is zero, A+B = B; when B is zero, A+B = A; and when A and B are zero, A+B = O.

Example 1. Required the value of $5_n\text{U}+12_p\text{U}$.

The nullifier of 5_nU and 12_pU is $(12-5)_n$U (Art. 6), which is the nullifier of $(12-5)_p$U, or 7_pU; ∴ 7_pU, being the quantity whose nullifier is that of 5_nU and 12_pU, is the value of $5_n\text{U}+12_p\text{U}$.

Example 2. Required the value of $4_p\text{U}+8_n\text{U}+7_p\text{U}$.

$4_p\text{U}+8_n\text{U}+7_p\text{U} = (8-4)_n\text{U}+7_p\text{U}$ (Ex. 1) $= [7-(8-4)]_p\text{U}$ (Ex. 1) $= 3_p\text{U}$. ∴ $4_p\text{U}+8_n\text{U}+7_p\text{U} = 3_p\text{U}$.

Cor. 2. $2_pU = U+U$; $3_pU = U+U+U$; $4_pU = U+U+U+U$; &c.

Thus, $4_pU = U+U+U+U$.

For, by (Ex. 1), $4_pU = 3_pU+1_pU = 2_pU+1_pU+1_pU = 1_pU+1_pU+1_pU+1_pU = U+U+U+U$.

I. $A+B = B+A$.

For $A+B$ and $B+A$ have each the same nullifier as A and B: but quantities which have the same nullifier are equal; $\therefore A+B = B+A$.

II. $A+B+C = A+C+B$.

The simplest case is that in which one or more of the quantities A, B, C is zero: we will take that in which they are all of real value. Let A, B, C have to any quantity U of the same kind, the respective ratios 5_n, 8_p, 12_n, so that A, B, C are equal respectively to 5_nU, 8_pU, 12_nU: then $5_nU+8_pU+12_nU = 5_nU+12_nU+8_pU$.

For, by (Ex. 2), $5_nU+8_pU+12_nU = [12-(8-5)]_nU$ and $5_nU+12_nU+8_pU = (12+5-8)_nU$: but $[12-(8-5)]_nU = (12+5-8)_nU$; therefore, $5_nU+8_pU+12_nU = 5_nU+12_nU+8_pU$.

The same kind of proof is applicable to other values of A, B, C.

8. In the propositions following we will suppose $a = 5_p$ and $b = 12_n$, bearing in mind that the same kind of proof is applicable to other values of a and b.

I. $aU + bU$ has a constant ratio to U for all real values of U.

For $aU + bU = 5_pU + 12_nU = (12-5)_nU$ (Art. 7); $\therefore$ the ratio of $aU + bU$ to U is $(12-5)_n$ for all real values of U.

II. $a(bU)$ has a constant ratio to U for all real values of U.

For, since bU is unlike to U, and $a(bU)$ is like to bU, it is evident that $a(bU)$ is unlike to U; also the numerical value of the ratio of $a(bU)$ to U is 5×12, as is evident; $\therefore$ the ratio of $a(bU)$ to U is the negative ratio $(5 \times 12)_n$ for all real values of U.

Cor. 1. The ratio of $a(bU)$ to U is positive or negative according as a is like or unlike to b, and equal numerically to the product of a and b.

Thus, the ratio of $5_p(12_nU)$ to U is $(5 \times 12)_n$ as above shown.

COR. 2. $a(bU) = b(aU)$.

For we may show as above that $a(bU) = (5\times 12)_n U$ and $b(aU) = (12\times 5)_n U$; but $(5\times 12)_n U = (12\times 5)_n U$; $\therefore a(bU) = b(aU)$.

9. DEFINITION. $a+b$ is the constant ratio of $aU+bU$ to U for all real values of U (I. Art. 8); so that $aU+bU = (a+b)U$.

COR. 1. When a is zero, $a+b = b$: when b is zero, $a+b = a$; and when a and b are zero $a+b = o$.

This is evident from (Art. 4, and Cor. 1, Art. 7).

COR. 2. $(a+b+c+d)U = aU+bU+cU+dU$.

For, by the Def., $(a+b+c+d)U = (a+b+c)U+dU = (a+b)U+cU+dU = aU+bU+cU+dU$.

NOTE. $o+a$ is abbreviated into $+a$; $1+1$ into 2; $1+1+1$ or $2+1$ into 3; $10+1$ into 11; $10+2$ into 12, &c., as in arithmetic.

I. $a+b = b+a$.

For since $(a+b)U = aU+bU$ (Def.) $= bU+aU$ (I., Art. 7) $= (b+a)U$ (Def.), we have $(a+b)U = (b+a)U$; $\therefore a+b = b+a$ (Art. 5, Cor. 2).

II. $a+b+c = a+c+b$.

For, since $(a+b+c)\mathrm{U} = a\mathrm{U}+b\mathrm{U}+c\mathrm{U}$ (Cor. 2) $= a\mathrm{U}+c\mathrm{U}+b\mathrm{U}$ (II., Art. 7) $= (a+c+b)\mathrm{U}$ (Cor. 2), we have $(a+b+c)\mathrm{U} = (a+c+b)\mathrm{U}$; $\therefore$ $a+b+c = a+c+b$ (Art. 5, Cor. 2).

10. All permutations of the same terms connected by the sign $+$ are equal to one another.

Thus, $a+b+c+d = c+b+d+a$.

For $a+b+c+d = b+a+c+d$ (I. Art. 9) $= b+c+a+d$ (II. Art. 9) $= b+c+d+a$ (II. Art. 9) $= c+b+d+a$ (I. Art. 9).

Cor. All combinations of the same terms connected by the sign $+$ are equal to one another.

Ex. 1. $a+(b+c+d) = a+b+c+d$.

For $a+(b+c+d) = b+c+d+a$ (I. Art. 9) $= a+b+c+d$ (Art. 10).

Ex. 2. $a+b+(c+d)+(e+f+g) = a+b+c+d+e+f+g$.

For $a+b+(c+d)+(e+f+g = a+b+c+d+(e+f+g)$ [Ex. 1] $= a+b+c+d+e+f+g$ [Ex. 1].

Ex. 3. $a+b+(c+d)+(e+f+g) = d+f+(e+g+b)+(a+c)$.

13. To add expressions such as $(b+c+d)$, $(-b+c-d+e)$, &c., remove the brackets, retaining the sign of each term.

Ex. 1. $a+(b+c+d)=a+b+c+d$ (Cor., Art. 10).

Ex. 2. $a+(-b+c-d+e) = a-b+c-d+e$.

For $a+(-b+c-d+e) = -b+c-d+e+a$ (I., Art. 9) $= a-b+c-d+e$ (Art. 12).

14. To subtract expressions such as $(b+c+d)$, $(-b+c-d+e)$, &c., remove the brackets and change the sign of each term.

Ex. 1. $a-(b+c+d) = a-b-c-d$.

For, since $a-b-c-d+(b+c+d) = a-b-c-d+b+c+d$ (Art. 13) $= a$ (Cor., Art. 12), we have $a-b-c-d+(b+c+d) = a$; $\therefore a-(b+c+d) = a-b-c-d$ (Cor. 2, Art. 11).

Ex. 2. $a-(-b+c-d+e) = a+b-c+d-e$.

For, since $a+b-c+d-e+(-b+c-d+e) = a+b-c+d-e-b+c-d+e$ (Art. 13) $= a$ (Cor., Art. 12), we have $a+b-c+d-e+(-b+c-d+e) = a$; $\therefore a-(-b+c-d+e) = a+b-c+d-e$ (Cor. 2, Art. 11).

Cor. 1. When a is zero, the above results become $-(b+c+d) = -b-c-d$; and $-(-b+c-d+e) = b-c+d-e$.

Cor. 2. An even number of negative signs before a symbol may be omitted.

Ex. 1. $-(-a)$ or $- -a = a$.

For $o-(-a) = o+a$ (Art. 14) $= a$ (Art. 9, Cor. 1).

Ex. 2. $- - - - - -a = -a$.

For $- -[- -(-a)] = - -(-a) = -a$ (Ex. 1).

15. We may now reduce compound expressions consisting of positive and negative terms to simpler or more convenient forms.

Thus, $a+b+(-c+d-e)+f-(g-h+k)-l = a+b-c+d-e+f-(g-h+k)-l$ (Art. 13) $= a+b-c+d-e+f-g+h-k-l$ (Art. 14) $= a+b+d+f+h-c-e-g-k-l$ (Art. 12) $= a+b+d+f+h-(c+e+g+k+l)$ (Art. 14).

Note. The terms numerical, integer, &c., will be applied to the symbols 1, 2, 3, &c., as defined in Art. 9, and to their opposites -1, -2, -3, &c. (Art. 11, Cor. 1); not to invest them with an arithmetical character, but to distinguish them from other symbols different in value or in form. No confusion can arise from the double use of these terms in an arithmetical and in an algebraical sense, if it be remembered that 1, 2, 3, &c., are never, except in the notation used in Art. 3, to be arithmetically understood.

Cor. When a, b, c, &c., are integers, the numerical equivalent of $a+b+c$, &c., is indicated by the arithmetical sum of a, b, c, &c.

Thus, $2+3+8 = 13$.

For, since 2, 3, 8 are abbreviations respectively of $1+1$, $1+1+1$, and $1+1+1+1+1+1+1+1$ (Art. 9), we have $2+3+8 = 1+1+(1+1+1)+(1+1+1+1+1+1+1+1) = 1+1+1+1+1+1+1+1+1+1+(1+1+1)$ (Art. 10) $= 10+3$ or 13.

The following are examples of the reduction of numerical expressions :—

Ex. 1. $5-9 = 5-(5+4) = 5-5-4 = -4$ (Art. 12, Cor).

Ex. 2. $-9+15 = -9+(9+6) = -9+9+6 = 6$.

Ex. 3. $-5-4-7 = -(5+4+7) = -16$.

Ex. 4. Required the numerical equivalent of $a+b+c$ when a, b, c are equal respectively to -6, 4, -9.

$a+b+c = -6+4+(-9) = 4-(6+9) = 4-15 = -11$ (Ex. 1).

Ex. 5. Required the numerical equivalent of $a-b+c-d$ when a, b, c, d are equal respectively to 4, -7, -6, and 3.

$a-b+c-d = 4-(-7)+(-6)-3 = 4+7-(6+3) = 11-9 = 2$.

16. **Definition.** $a \times b$ or ab is the constant ratio of $a(b\mathrm{U})$ to U for all real values of U (II., Art. 8); so that $a(b\mathrm{U}) = ab.\mathrm{U}$.

Cor. When either a or b is zero, $ab = o$ (Art. 4). Also when $a = 1$, $ab = b$; and when $b = 1$, $ab = a$ (Art. 5).

I. $ab = ba$.

For, since $ab.\mathrm{U} = a(b\mathrm{U})$ (Def.) $= b(a\mathrm{U})$ (II., Cor. 2, Art. 8) $= ba.\mathrm{U}$ (Def.), we have $ab.\mathrm{U} = ba.\mathrm{U}$; $\therefore$ $ab = ba$ (Art. 5, Cor. 2).

II. $abc = acb$.

For, since $abc.\mathrm{U} = ab(c\mathrm{U})$ (Def.) $= a[b(c\mathrm{U})]$ (Def.) $= a[c(b\mathrm{U})]$ (II., Cor. 2, Art. 8) $= ac(b\mathrm{U})$ (Def.) $= acb.\mathrm{U}$ (Def.), we have $abc.\mathrm{U} = acb\mathrm{U}$; $\therefore$ $abc = acb$ (Art. 5, Cor. 2).

III. $(b+c)a = ba+ca$.

For, since $[(b+c)a].\mathrm{U} = (b+c)(a\mathrm{U})$ (Def.) $= b(a\mathrm{U})+c(a\mathrm{U})$ (Def., Art. 9) $= ba.\mathrm{U}+ca.\mathrm{U}$ (Def.) $= (ba+ca)\mathrm{U}$ (Def., Art. 9), we have $[(b+c)a].\mathrm{U} = (ba+ca).\mathrm{U}$; $\therefore$ $(b+c)a = ba+ca$ (Art. 5, Cor. 2).

17. All combinations of the same factors are equal to one another.

See the proof of Art. 10.

18. The rule of signs in the multiplication of simple terms is as follows; viz.:—

$$-a\times b = a\times(-b) = -ab, \text{ and } -a\times(-b) = ab.$$

1st, $-a\times b = -ab$.

For, since $o = (-a+a).b$ (Def., Art. 11, and Cor., Art. 16) $= (-a\times b)+ab$ (III., Art. 16), we have $-a\times b = o-ab$ (Art. 11, Cor. 2).

2dly, $a\times(-b) = -ab$.

For $a\times(-b) = -b\times a$ (I., Art. 16) $= -ba$ (Case 1) $= -ab$ (I., Art. 16).

3dly, $-a\times(-b) = ab$.

For $-a\times(-b) = -[a\times(-b)]$ (Case 1) $= -(-ab)$ (Case 2) $= ab$ (Art. 14, Cor. 2).

Cor. The sign of a product is negative or positive, according as the number of its negative factors is odd or even.

Ex. 1. $a\times(-b)\times c = -abc$.

For $a\times(-b)\times c = -ab\times c = -abc$.

Ex. 2. $-a\times b\times c\times(-d) = abcd$.

For $-a\times b\times c\times(-d) = -ab\times c\times(-d) = -abc\times(-d) = abcd$.

19. $(b+c+d+e).a = ba+ca+da+ea.$

For $(b+c+d+e).a = (b+c+d)a+ea = (b+c)a + da+ea = ba+ca+da+ea$ (III., Art. 16).

Cor. 1. The order of the factors in the above result may be changed (I., Art. 16).

Cor. 2. When a, b, c, &c., are integers, the numerical equivalent of $a\times b\times c$, &c., is indicated by the arithmetical product of a, b, c, &c. Thus, $5\times 7 = 35$.

For, since 5 is an abbreviation of $1+1+1+1+1$ (Art. 9), we have $5\times 7 = (1+1+1+1+1)\times 7 = (1\times 7)+(1\times 7)+(1\times 7)+(1\times 7)+(1\times 7)$ (Art. 19) $= 7+7+7+7+7$ (Cor., Art. 16) $= 35$ (Art. 15).

20. $(-b-c+d-e+f)a = -ba-ca+da-ea+fa.$

For $(-b-c+d-e+f)a = [(-b)+(-c)+d+(-e)+f].a$ (Art. 11, Cor. 4) $= (-b\times a)+(-c\times a)+da+(-e\times a)+fa$ (Art. 19) $= -ba+(-ca)+da(-ea)+fa$ (Art. 18) $= -ba-ca+da-ea+fa$ (Art. 11, Cor. 4).

Cor. The order of the factors in the above result may be changed (I., Art. 16).

21. **Definition.** The value of $a \div b$ is that which makes $a \div b \times b = a$.

Note. $a \div b$ is usually written $\frac{a}{b}$.

Cor. 1. If $d \times b = a$, then $a \div b = d$.*

For, since $d \times b = (a \div b) \times b$, each being equal to a, multiply each by $1 \div b$ and transpose the factors (Art. 17), which gives $(1 \div b) \times b \times d = (1 \div b) \times b \times (a \div b)$ or $1 \times d = 1 \times (a \div b)$, because $(1 \div b) \times b = 1$ (Def.); $\therefore$ $a \div b = d$ (Cor., Art. 16).

Cor. 2. When $a = o$, $a \div b = o$; when $b = 1$, $a \div b = a$; and when $a = b$, $a \div b = 1$.

For, when $a = o$, we have $o \times b = a$ (Cor., Art. 16) $\therefore$ $a \div b = o$. When $b = 1$, we have $a \times b = a$ (Cor., Art. 16) $\therefore$ $a \div b = a$. And when $a = b$, we have $1 \times b = a$ (Cor., Art. 16) $\therefore$ $a \div b = 1$.

22. We can now assign their proper meanings to b and $-b$, when b is exhibited in the form of a numerical integer or fraction.

1, 2, 3, 4, &c., are positive ratios, whose numerical values are one, two, three, four, &c. Thus $4 = 4_p$ (Art. 3).

For, since $4U = (1+1+1+1)U$ (Note, Art. 9) $= 1U+1U+1U+1U$ (Art. 9, Cor. 2) $= U+U+U+U$ (Art. 5) $= 4_pU$ (Art. 7, Cor. 2), we have $4U = 4_pU$: $\therefore\ 4 = 4_p$ (Art. 5, Cor. 2).

$\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$, &c., are positive ratios, whose numerical values are one-half, two-thirds, five-sixths, &c. Thus, $\frac{5}{6} = (\frac{5}{6})_p$.

For, since $(\frac{5}{6})_p \times 6 = (\frac{5}{6})_p \times 6_p = 5_p$ (II., Cor. 1, Art. 8, and Def., Art. 16) $= 5$, we have $(\frac{5}{6})_p \times 6 = 5$; $\therefore\ \frac{5}{6} = (\frac{5}{6})_p$ (Art. 21, Cor. 1).

It is proved, in Art. 11, Cor. 1, that when *b* is real, $-b$ is numerically equal, but unlike to *b*. Hence—

-1, -2, -3, &c., are negative ratios, whose numerical values are one, two, three, &c.

And $-\frac{1}{2}$, $-\frac{2}{3}$, $-\frac{5}{6}$, &c., are negative ratios, whose numerical values are one-half, two-thirds, five-sixths, &c.

23. The rule of signs in the division of simple terms is as follows; viz.:—

$a \div (-b) = -a \div b = -(a \div b)$; and $-a \div (-b) = a \div b$.

Thus, $a \div (-b) = -(a \div b)$.

For, since $-(a \div b) \times (-b) = a \div b \times b$ (Art. 18) $= a$ (Art. 21), we have $-(a \div b) \times (-b) = a$; $\therefore\ a \div (-b) = -(a \div b)$ (Art. 21, Cor. 1).

The other cases may be similarly proved.

24. $\dfrac{b+c+d+e}{a} = \dfrac{b}{a} + \dfrac{c}{a} + \dfrac{d}{a} + \dfrac{e}{a}$.

For, since $\left(\dfrac{b}{a} + \dfrac{c}{a} + \dfrac{d}{a} + \dfrac{e}{a}\right) \times a = \left(\dfrac{b}{a} \times a\right) + \left(\dfrac{c}{a} \times a\right) + \left(\dfrac{d}{a} \times a\right) + \left(\dfrac{e}{a} \times a\right)$ (Art. 19) $= b+c+d+e$ (Art. 21), we have $\left(\dfrac{b}{a} + \dfrac{c}{a} + \dfrac{d}{a} + \dfrac{e}{a}\right) \times a = b+c+d+e$; $\therefore\ \dfrac{b+c+d+e}{a} = \dfrac{b}{a} + \dfrac{c}{a} + \dfrac{d}{a} + \dfrac{e}{a}$ (Cor. 1, Art. 21).

25. $\dfrac{-b+c-d+e}{a} = -\dfrac{b}{a} + \dfrac{c}{a} - \dfrac{d}{a} + \dfrac{e}{a}$.

For, since $\left(-\dfrac{b}{a} + \dfrac{c}{a} - \dfrac{d}{a} + \dfrac{e}{a}\right) \times a = -\left(\dfrac{b}{a} \times a\right) + \left(\dfrac{c}{a} \times a\right) - \left(\dfrac{d}{a} \times a\right) + \left(\dfrac{e}{a} \times a\right)$ (Art. 20) $= -b+c$

$-d+e$ (Art. 21), we have $\left(-\frac{b}{a}+\frac{c}{a}-\frac{d}{a}+\frac{e}{a}\right)\times a$ $=-b+c-d+e$; $\therefore\ \frac{-b+c-d+e}{a}=-\frac{b}{a}+\frac{c}{a}-\frac{d}{a}$ $+\frac{e}{a}$ (Cor. 1, Art. 21).

Or thus,

$\frac{-b+c-d+e}{a}=\frac{(-b)+c+(-d)+e}{a}$ (Art. 11, Cor. 4) $=\frac{-b}{a}+\frac{c}{a}+\frac{-d}{a}+\frac{e}{a}$ (Art. 24) $=$ $-\frac{b}{a}+\frac{c}{a}+\left(\frac{-d}{a}\right)+\frac{e}{a}$ (Art.23) $=-\frac{b}{a}+\frac{c}{a}-\frac{d}{a}+\frac{e}{a}$ (Art. 11, Cor. 4).

26. The numerator and denominator of an algebraical fraction may both be multiplied or divided by the same factor of real value.

Thus, $\frac{b}{a}=\frac{ab}{ac}$.

For, since $\frac{b}{c}\times ac=a\times\left(\frac{b}{c}\times c\right)$ (Art. 17) $=ab$ (Art. 21), we have $\frac{b}{c}\times ac=ab$; $\therefore\ \frac{b}{c}=\frac{ab}{ac}$ (Art. 21, Cor. 1).

Hence $\frac{b}{c}=\frac{b\div a}{c\div a}$.

For $\dfrac{b \div a}{c \div a} = \dfrac{b \div a \times a}{c \div a \times a} = \dfrac{b}{c}$ (Art. 21).

Cor. An algebraical fraction may be reduced to lower terms by resolving the numerator and denominator into factors, and omitting those common to both.

Ex. 1. $\dfrac{ceag}{egc} = a.$

For $\dfrac{ceag}{egc} = \dfrac{egc \times a}{egc \times 1}$ (Art. 17 and Cor., Art. 16) $= \dfrac{a}{1}$ (Art. 26) $= a$ (Cor. 2, Art. 21).

Ex. 2. $\dfrac{ab \times cde}{b \times df} = \dfrac{ace}{f}.$

For $\dfrac{ab \times cde}{b \times df} = \dfrac{bd \times ace}{bd \times f}$ (Art. 17) $= \dfrac{ace}{f}$ (Art. 26).

Ex. 3. $\dfrac{18ab}{117ad} = \dfrac{2b}{13d}.$

For $\dfrac{18ab}{117ad} = \dfrac{9 \times 2 \times a \times b}{9 \times 13 \times a \times d}$ (Art. 19, Cor. 2) $= \dfrac{2b}{13d}$ (Ex. 2).

27. Algebraical fractions may be changed to others of equal values with a common denominator.

Ex. 1. $a, \frac{b}{c}, \frac{d}{e}, \frac{f}{g}$ are equal respectively to $\frac{aceg}{ceg}, \frac{beg}{ceg}, \frac{cdg}{ceg}, \frac{cef}{ceg}$.

For, if a be treated as $\frac{a}{1}$ (Cor. 2, Art. 21), and the numerator and denominator of each fraction be multiplied by the product of all the denominators except its own (Art. 26), we have by (Cor., Art. 26), $a = \frac{aceg}{ceg}, \frac{b}{c} = \frac{beg}{ceg}$, &c.

Ex. 2. $\frac{a}{b}, c, \frac{d}{e}$, and $\frac{f}{b}$ are equal respectively to $\frac{ae}{be}, \frac{bce}{be}, \frac{bd}{be}$; and $\frac{ef}{be}$.

For, if we multiply the denominator of each fraction by a factor which reduces it to be which is the lowest common multiple of all the denominators, and at the same time multiply the numerator by the same factor (Art. 26), we have $\frac{a}{b} = \frac{ae}{be}, c = \frac{bce}{be}$, &c. (Cor., Art. 26).

Ex. 3. $5, \frac{6}{7}, \frac{3}{4}, \frac{1}{2}$, are equal respectively to $\frac{140}{28}, \frac{24}{28}, \frac{21}{28}, \frac{14}{28}$.

For, if $5, \frac{6}{7}, \frac{3}{4}, \frac{1}{2}$ be written in the form $\frac{5}{1}, \frac{6}{7}, \frac{3}{2 \times 2}, \frac{1}{2}$ (Art. 21, Cor. 2, and Art. 19, Cor. 2);

then $7 \times 2 \times 2$ being the lowest common multiple of all the denominators, we may show, as in Ex. 2, that $5 = \frac{5. \times 7 \times 2 \times 2}{1 \times 7 \times 2 \times 2} = \frac{140}{28}$, $\frac{6}{7} = \frac{6 \times 2 \times 2}{7 \times 2 \times 2} = \frac{24}{28}$, &c.

Ex. 4. $\frac{a}{2b}$, 13, $\frac{7}{18c}$, $\frac{4d}{5b}$ are equal respectively to $\frac{45ac}{90bc}$, $\frac{1170bc}{90bc}$, $\frac{35b}{90bc}$, $\frac{72cd}{90bc}$; the literal factors being reduced as in Ex. 2, and the numerical factors as in Ex. 3.

28. The addition and subtraction of algebraical factions may be effected by reducing them to fractions of equal values with a common denominator (Art. 27), and collecting the numerators of the reduced fractions into one numerator, by Art. 24 and 25.

Ex. 1. $a + \frac{b}{c} + \frac{d}{e} + f = \frac{ace}{ce} + \frac{be}{ce} + \frac{cd}{ce} + \frac{cef}{ce}$ $= \frac{ace + be + cd + cef}{ce}$.

Ex. 2. $a - \frac{b}{c} + \frac{d}{e} - f = \frac{ace}{ce} - \frac{be}{ce} + \frac{cd}{ce} - \frac{cef}{ce}$ $= \frac{ace - be + cd - cef}{ce}$.

Ex. 3. $\frac{5}{6} + 4 + \frac{7}{10} + 8 = \frac{25}{30} + \frac{120}{30} + \frac{21}{30} +$

$+\frac{240}{30}=\frac{25+120+21+240}{30}=\frac{406}{30}$ (Cor., Art. 15) $=\frac{203}{15}$ (Cor., Art. 26).

Ex. 4. $\frac{3}{2}-5+\frac{5}{6}-\frac{7}{8}+\frac{2}{3}=\frac{36}{24}-\frac{120}{24}+\frac{20}{24}-\frac{21}{24}+\frac{16}{24}=\frac{36-120+20-21+16}{24}=\frac{-69}{24}$ (Cor., Art. 15) $=-\frac{69}{24}$ (Art. 23) $=-\frac{23}{8}$ (Cor., Art. 26).

29. The rules for multiplication and division of algebraical fractions are as follow:—

I. $a\times\frac{c}{d}=\frac{ac}{d}$; $\frac{a}{b}\times c=\frac{ac}{b}$; $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

Thus $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

For, since $\frac{a}{b}\times\frac{c}{d}\times bd=\left(\frac{a}{b}\times b\right)\times\left(\frac{c}{d}\times d\right)$ (Art. 17) $=ac$ (Def., Art. 21), we have $\frac{a}{b}\times\frac{c}{d}\times bd=ac$;

$\therefore\ \frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$ (Cor. 1, Art. 21).

The other cases may be similarly proved.

II. $a \div \frac{c}{d} = \frac{ad}{c}$; $\frac{a}{b} \div c = \frac{a}{bc}$; $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$.

Thus, $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$.

For, since $\frac{ad}{bc} \times \frac{c}{d} = \frac{adc}{bcd}$ (I.) $= \frac{a}{b}$ (Cor., Art. 26), we have $\frac{ad}{bc} \times \frac{c}{d} = \frac{a}{b}$; $\therefore\ \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ (Cor. 1, Art. 21).

The other cases may be similarly proved.

Ex. 1. $\frac{2}{5} \times 6 \times \frac{7}{8} = \frac{2 \times 6}{5} \times \frac{7}{8} = \frac{2 \times 6 \times 7}{5 \times 8}$ $= \frac{2 \times (2 \times 3) \times 7}{5 \times (2 \times 2 \times 2)}$ (Art. 19, Cor. 2) $= \frac{3 \times 7}{5 \times 2}$ (Cor., Art. 26) $= \frac{21}{10}$.

Ex. 2. $\frac{30}{7} \div \frac{3}{35} = \frac{30 \times 35}{7 \times 3} = \frac{3 \times 10 \times (7 \times 5)}{7 \times 3}$ (Art. 19, Cor. 2) $= 10 \times 5$ (Cor., Art. 26) $= 50$.

30. Expressions consisting of positive and negative terms, with numerical coefficients either whole or fractional, may be reduced as follows:—

Ex. 1. $2a + 3a + 6a + a = (2 + 3 + 6 + 1)a$ (Art. 19 and Cor., Art. 16) $= 12a$ (Cor., Art. 15).

Ex. 2. $3a-5a=(3-5)a$ (Art. 20) $=-2\times a$ (Cor., Art. 15) $=-2a$ (Art. 18).

Ex. 3. $2a-b-(-c+3a-b)+4c=(2a-3a)+(c+4c)$ (Art. 15) $=-a+5c$ (Ex. 1 and 2).

Ex. 4. $\frac{3}{4}a-b+\frac{2}{3}c+5a-\frac{6}{7}c=(\frac{3}{4}a+5a)+(\frac{2}{3}c-\frac{6}{7}c)-b$ (Art. 15) $=(\frac{3}{4}+5)a+(\frac{2}{3}-\frac{6}{7})c-b$ (Art. 19 and 20) $=\frac{23}{4}a+(-\frac{4}{21})c-b$ (Art. 28) $=\frac{23}{4}a+(-\frac{4}{21}c)-b$ (Art. 18) $=\frac{23}{4}a-\frac{4}{21}c-b$ (Cor. 4, Art. 11).

31. The following are examples of simple multiplication :—

Ex. 1. $3a\times(-4b)\times 6c=-(3a\times 4b\times 6c)$ (Cor., Art. 18) $=-(3\times 4\times 6\times abc)$ (Art. 17) $=-72abc$ (Cor. 2, Art. 19).

Ex. 2. $\frac{2}{3}a\times\frac{5}{6}b\times(-\frac{7}{8}c)=-(\frac{2}{3}a\times\frac{5}{6}b\times\frac{7}{8}c)$ (Cor., Art. 18) $=-(\frac{2}{3}\times\frac{5}{6}\times\frac{7}{8}\times abc)$ (Art. 17) $=-\frac{35}{72}abc$ (Art. 29).

Ex. 3. $(-2+3b-5c)\times 6d=(-2\times 6d)+(3b\times 6d)-(5c\times 6d)$ (Art. 20) $=-12d+18bd-30cd$ (Ex. 1).

Ex. 4. $(5ad-\frac{6}{7}bc-\frac{2}{3}ac)\times -\frac{3}{4}bd=(5ad\times -\frac{3}{4}bd)-(\frac{6}{7}bc\times -\frac{3}{4}bd)-(\frac{2}{3}ac\times -\frac{3}{4}bd)$ (Art. 20) $=$

$-\frac{15}{4}abdd-(-\frac{9}{14}bbcd)-(-\frac{1}{2}abcd)$ (Ex. 2) $=$ $-\frac{15}{4}abdd+\frac{9}{14}bbcd+\frac{1}{2}abcd$ (Art. 15).

32. The following are examples of simple division: –

Ex. 1. $\frac{4abc}{2ac} = \frac{2ac\times 2b}{2ac}$ (Art. 31) $= 2b$ (Cor., Art. 26).

Ex. 2. $\frac{\frac{5}{6}acb}{\frac{3}{4}ab} = \frac{\frac{5}{6}}{\frac{3}{4}}\times\frac{acb}{ab}$ (I., Art. 29) $=$ $\frac{10}{9}\times\frac{acb}{ab}$ (II., Art. 29) $= \frac{10}{9}c$ (Ex. 1).

Ex. 3. $\frac{-2ab+2bc-4abc}{2b} = \frac{-2ab}{2b}+\frac{2bc}{2b}$ $-\frac{4abc}{2b}$ (Art. 25) $= -a+c-2ac$ (Ex. 1).

Ex. 4. $\frac{\frac{5}{6}ac+\frac{3}{4}cd-\frac{2}{3}cc}{2c} = \frac{\frac{5}{6}ac}{2c}+\frac{\frac{3}{4}cd}{2c}-\frac{\frac{2}{3}cc}{2c}$ (Art. 25) $= \frac{5}{12}a+\frac{3}{8}d-\frac{1}{3}c$ (Ex. 2).

33. We may now find the products of expressions consisting of any number of positive and negative terms.

Ex. 1. $(a+b)(c+d) = (a+b)c+(a+b)d$ (Art. 19) $= (ac+bc)+(ad+bd)$ (Art. 19) $= ac+bc+ad$ $+bd$ (Art. 13).

Ex. 2. $(a-b)(c-d) = (a-b)(-d+c)$ (Art. 12) $= (a-b)c+(a-b)\times -d$ (Art. 19) $= (ac-bc)+(-ad+bd)$ (Art. 31) $= ac-bc-ad+bd$ (Art. 13).

Ex. 3. $(2a-b)(-a+3b) = (2a-b)\times -a+(2a-b)\times 3b$ (Art. 19) $= (-2aa+ab)+(6ab-3bb)$ (Art. 31) $= -2aa+7ab-3bb$ (Art. 30).

From these and similar examples we may deduce the following rule:—

Rule.—Supply (mentally at least) the sign $+$ where a sign is wanting; multiply the several terms of the multiplicand by each term of the multiplier, prefixing $+$ to the products of terms with like signs, and $-$ to the products of terms with unlike signs; add the partial products so obtained, making all possible reductions as in Art. 30.

In applying the above rule the process may be conducted as follows:—

$$\begin{array}{lcrl}
 & & 2a-b & \text{Multiplicand.}\\
 & & -a+3b & \text{Multiplier.}\\ \cline{3-3}
(2a-b)\times -a & = & -2aa+ab & \multirow{2}{*}{$\left.\begin{array}{c}\\ \\ \end{array}\right\}\text{Products to be added.}$}\\
(2a-b)\times 3b & = & 6ab-3bb & \\ \cline{3-3}
(2a-b)\times(-a+3b) & = & -2aa+7ab-3bb & \left\{\text{Product required.}\right.\\ \cline{3-3}
\end{array}$$

which is the result of Ex. 3.

For convenience of addition, the terms of the multiplier and multiplicand should be so arranged that the partial products which contain the same letters may fall in the same vertical line; thus, $6ab$ in the above example stands under ab.

34. To find the quotient of compound expressions:—

Let A stand for the dividend, and B for the divisor. Assume a convenient term Q, and let $B \times Q$ subtracted from A leave a remainder R; assume another term Q', and let $B \times Q'$ subtracted from R, leave a remainder R'; and suppose this process to be continued until there is no remainder; thus—

$$\begin{array}{lcl} B) & A & (Q \\ & BQ & \\ \cline{2-2} B) & R & (Q' \\ & BQ' & \\ \cline{2-2} B) & R' & (Q'' \\ & BQ'' & \\ \cline{2-2} & " & \\ \cline{2-2} \end{array}$$

Then the quotient of A by B is the sum of the successive quotients Q, Q', Q''; that is, $A \div B = Q + Q' + Q''$.

$$\begin{aligned} \text{For, since } A &= BQ + R \\ &= BQ + BQ' + R' \text{ because } R = BQ' + R'. \\ &= BQ + BQ' + BQ'' \text{ because } R' = BQ''. \\ &= B(Q + Q' + Q'') \end{aligned}$$

$$\text{we have } A = B(Q + Q' + Q''); \therefore A \div B = Q + Q' + Q''.$$

Example. Divide $aa + aab + abb - 2bb$ by $a + b$.

Assuming the successive quotients such that one term at least of each expression subtracted from disappears in the remainder, we have—

$$
\begin{array}{rl}
a+b\ ) & aa+aab+abb-2bb\ (\ a \\
 & \underline{aa+ab\qquad\qquad\qquad} \\
a+b\ ) & -ab+aab+abb-2bb\ (\ -b \\
 & \underline{-ab-bb\qquad\qquad\qquad} \\
a+b\ ) & aab+abb-bb\ (\ ab \\
 & \underline{aab+abb\qquad\quad} \\
a+b\ ) & -bb\ \left(-\frac{bb}{a+b}\right. \\
 & \underline{-bb} \\
 & \underline{\ \ ''\ \ }
\end{array}
$$

∴ The quotient is $a+(-b)+ab+\left(-\frac{bb}{a+b}\right) = a-b+ab-\frac{bb}{a+b}$.

In practice it is unnecessary to repeat the divisor before each remainder; the terms of the quotient may be arranged side by side at the right of the dividend as they occur; and in most cases the operation may be abridged by writing down as many terms only in each remainder as there are terms in the next product to be subtracted.

35. DEFINITION. When $a-b$ is positive in value, a is said to be *greater* than b.

Greater and *less* are correlative terms, b being less than a when a is greater than b, and *vice versa.* Hence—

COR. 1. a is greater than, or equal to, or less than b, according as $a—b$ is positive, or zero, or negative.

For, when $a-b$ is positive, a is greater than b by the definition; when $a-b = o$ then $a-b+b = o+b$ or $a = b$; and when $a-b$ is negative then $-(a-b)$ or $b-a$ is positive (Art. 11 Cor. 1) and b is greater than a, or a is less than b, by the definition.

COR. 2. All positive values are greater than zero; and all negative values are less than zero. Of two unlike values the positive value is the greater. Of two positive values that is the greater which is arithmetically the greater; and of two negative values that is the greater which is arithmetically the less.

1st. All positive values are greater than zero; and all negative values are less than zero.

Thus, 5 is greater than o because $5-o = 5$ is positive; -5 is less than o because $-5-o = -5$ is negative (Cor. 1).

2dly. Of two unlike values, the positive value is the greater.

Thus, 5 is greater than -7, because $5-(-7) = 5+7 = 12$ is positive.

3dly. Of two positive values, that is the greater which is arithmetically the greater; and of two negative values, that is the greater which is arithmetically the less.

Thus, 4 is greater than 2 because $4-2=2$ is positive; -3 is greater than -6 because $-3-(-6)=6-3=3$ is positive.

COR. 3. When a is greater than b and b is greater than c, then a is greater than c.

For since, by the Def., $a-b$ and $b-c$ are both positive, their sum, which is $a-c$, is positive; $\therefore$ a is greater than c (Def.)

I. $a+c$ is greater than $b+c$ when a is greater than b.

For, since $a-b$ is positive, its equal $(a+c)-(b+c)$ is positive; $\therefore$ $a+c$ is greater than $b+c$ (Def.)

II. $a-c$ is greater than $b-c$ when a is greater than b.

For, since $a-b$ is positive, its equal $(a-c)-(b-c)$ is positive; $\therefore$ $a-c$ is greater than $b-c$ (Def.)

III. $a-b$ is less than $a-c$ when b is greater than c.

For, since $b-c$ is positive, its equal $(a-c)-(a-b)$ is positive; $\therefore$ $a-c$ is greater than $a-b$ (Def.) or $a-b$ is less than $a-c$.

IV. ac is greater than bc when c is positive and a is greater than b.

For, since the factors c and $a-b$ are positive, their product which is $ac-bc$ is positive; $\therefore$ ac is greater than bc (Def.)

Cor. When c is negative, the inequality is reversed.

For, by changing the sign of c we change $(a-b)c$ (Art. 18) or $ac-bc$ from positive to negative, which makes ac less than bc (Art. 35, Cor. 1).

V. $a \div c$ is greater than $b \div c$, when c is positive, and a is greater than b.

For, since $a-b$ and c are positive, their quotient which is $(a \div c)-(b \div c)$ is positive; $\therefore$ $a \div c$ is greater than $b \div c$.

Cor. When c is negative the inequality is reversed.

For, by changing the sign of c we change $(a-b) \div c$ (Art. 23) or $(a \div c)-(b \div c)$ from positive to negative, which makes $a \div c$ less than $b \div c$ (Art. 35, Cor. 1).

VI. $a \div b$ is less than $a \div c$ when a is positive, and b is greater than and like to c.

For, since a, $b-c$, and bc (Art. 18) are all positive, the quotient $a(b-c) \div bc$ (Art. 18 and 23) or

$(a \div c) - (a \div b)$ is positive; $\therefore$ $a \div c$ is greater than $a \div b$ (Def.) or $a \div b$ is less than $a \div c$.

COR. When a is negative, or when b is unlike to c, the inequality is reversed.

For, in either case, we change $a(c-b) \div bc$ (Art. 18 and 23) or $(a \div b) - (a \div c)$ from negative to positive, which makes $a \div b$ greater than $a \div c$.

NOTE.—All the rest of the Algebra of Positive and Negative Quantities may in the usual manner be deduced from the principles and processes which have been now explained. I have, therefore, in the next articles supposed the reader to be acquainted with notation and rules of operation gathered from other and larger works.

36. The sign of inequality is $>$, the point being turned towards the greater quantity. Thus $5 > -7$ or $-7 < 5$ denotes that 5 is greater than -7 or that -7 is less than 5.

Since d^2 is essentially positive for all real and possible values of d, an inequality $a > b$ may be symbolically expressed by the equation $a - b = d^2$ or by $a = b + d^2$.

So also to express symbolically that a quantity b is actually increased, we must add to it not d but d^2.

The principles in the solution of inequalities which correspond to those in the solution of equations are the following :—

I. If $a > b$ then $a \pm c > b \pm c$ (I. and II., Art. 35).

Cor. 1. Any quantity may be transposed from one side of an inequality to the other with its sign changed.

Thus, if $2x - 9 > 15 + x$ then
adding 9 to each side $2x > 15 + 9 + x$ and
subtracting x from each side $2x - x > 15 + 9$
or $x > 24$

which solution has been effected by transposing the terms -9 and x to the opposite side of the inequality with their signs changed.

Cor. 2. If any quantity be found with the same sign on both sides of an inequality it may be omitted in each.

Thus, if $x + y > 5 + y$ then
by transposition $x + y - y > 5$
or $x > 5$.

Cor. 3. When known and unknown quantities are connected in an inequality by the signs + and —, they may be separated by transposing the known quantities to one side and the unknown quantities to the other.

Thus, if $3x^2 - 4 + 2x > x^2 - 5x + 8$ then
by transposition $3x^2 - x^2 + 2x + 5x > 8 + 4$
or $2x^2 + 7x > 12$.

II. If $a > b$ then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$ for all *positive* values of c (IV. and V., Art. 35).

COR. 1. An inequality may be cleared of fractional terms by multiplying each side by any *positive* common multiple of all the denominators on both sides; the lowest common multiple being most convenient.

Thus, if $\frac{x}{2} + \frac{x}{3} > 13 + \frac{x}{4} + \frac{5}{12}$ then multiplying each side by 12, which is the least common multiple of 2, 3, 4, 12, we have—

$$6x + 4x > 156 + 3x + 5, \text{ and}$$

by transposition $6x + 4x - 3x > 156 + 5$

$$\text{or } 7x > 161$$

$\therefore$ dividing each side by 7 $x > 23$

COR. 2. An inequality may be reduced to lower terms by dividing each side by any *positive* common measure of the two.

The principles in the solution of inequalities which do not correspond to those in the solution of equations are the following:—

III. If unequal quantities be subtracted from the same or equal quantities, the sign of inequality must be changed.

Thus, if $b > c$ then $a - b < a - c$ (III., Art. 35).

Cor. 1. If $a > b$ then $-a < -b$.

Hence we cannot (as in equations) change the signs of all the terms of an inequality, without changing the sign of inequality.

Thus, if $a^2 - b^2 > 2ab - c^2$ then
$-(a^2 - b^2) < -(2ab - c^2)$ as above proved
or $b^2 - a^2 < c^2 - 2ab$

in which the signs of all terms of the given inequality are reversed, but the sign of inequality is changed.

The same result follows from I., Cor. 1; for, if each term of the inequality $a^2 - b^2 > 2ab - c^2$ be transposed to the opposite side with its sign changed, we have $c^2 - 2ab > b^2 - a^2$.

IV. Unequal quantities cannot be multiplied or divided by the same *negative* quantity, without changing the sign of inequality.

Thus, if $a > b$ and c be negative, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$ (IV. Cor. and V. Cor., Art. 35).

Cor. An inequality cannot (as in equations) be cleared of fractions or reduced to lower terms by multiplying or dividing each side by the same *negative* quantity without changing the sign of inequality.

V. Inequalities, which are true of given quantities, are *not always* true of the same powers of those quantities.

Thus, if $a > b$, it is not always true that $a^m > b^m$ or that $a^{\frac{1}{m}} > b^{\frac{1}{m}}$.

For, though $2 > -4$, it is not true that $2^2 > (-4)^2$ or that $4 > 16$; and, conversely, though $2^2 < (-4)^2$, it is not true that $2 < -4$.

Cor. When an inequality has been reduced to the form $x^m > a$, we cannot, except when positive values only are concerned, conclude certainly that $x > \sqrt[m]{a}$.

Thus, $(-2)^2 > 3$, but it is not true that $-2 > \sqrt{3}$.

Neither can we safely (as in equations), except when positive values only are concerned, clear an inequality of surds or terms under the radical sign by involution of each side.

Ex. 1. If $4x-7 < 2x+3$ and $3x+1 > 13-x$ find an integral value of x.

$$\begin{array}{cc} 4x-7 < 2x+3 \text{ and } & 3x+1 > 13-x \\ 4x-2x < 7+3 & 3x+x > 13-1 \\ 2x < 10 & 4x > 12 \\ x < 5 & x > 3 \end{array}$$

$\therefore$ x being an integer greater than 3 and less than 5 is equal to 4.

Ex. 2. Which is greater $\sqrt{11}+\sqrt{7}$ or $\sqrt{19}+\sqrt{2}$.

Squaring each side because both are positive (V., Cor.)

$\sqrt{11}+\sqrt{7}$ is $>$ or $=$ or $<$ $\sqrt{19}+\sqrt{2}$ according as
$18+2\sqrt{77}$ .... $21+\sqrt{38}$ or as
$2\sqrt{77}$ .... $3+\sqrt{38}$ or (squaring each side because both are positive) according as
308 is $>$ or $=$ or $<47+6\sqrt{38}$ or as
261 .... $6\sqrt{38}$ or as
$43\frac{1}{2}$................ $\sqrt{38}$
but $43\frac{1}{2}$ is evidently greater than $\sqrt{38}$; $\therefore$ accordingly $\sqrt{11}+\sqrt{7}>\sqrt{19}+\sqrt{2}$.

Ex. 3. Which is greater, $\sqrt{a^2-b^2}+\sqrt{a^2-(a-b)^2}$ or a, when b is positive and $a>b$.

$\sqrt{a^2-b^2}+\sqrt{a^2-(a-b)^2}$ is $>$ or $=$ or $<a$
according as $\sqrt{a^2-(a-b)^2}$ .... $a-\sqrt{a^2-b^2}$
or (squaring each side because both are positive) according as—
$2ab-b^2$ is $>$ or $=$ or $<b^2-2a\sqrt{a^2-b^2}$ or as
$2(a-b)b$ .... $-2a\sqrt{a^2-b^2}$
but $2(a-b)b$ being positive is greater than $-2a\times\sqrt{a^2-b^2}$, which is negative; $\therefore$ accordingly $\sqrt{a^2-b^2}+\sqrt{a^2-(a-b)^2}>a$.

NOTE. $a-\sqrt{a^2-b^2}$, the quantity squared in the above reduction is evidently positive, being greater than $a-\sqrt{a^2}$ or o; b^2 being less than a^2, because b is positive and less than a.

37. Questions on inequality, in which two values only are compared, may often most readily be answered by resolving the difference of the two values into a product, the sign of which may be decided by the number and signs of its component factors (Art. 18), and the nature of the inequality then determined by the definition in Art. 35.

Ex. 1. x^2+y^2 cannot be less than $2xy$.

For $(x^2+y^2)-2xy = x^2-2xy+y^2$
$= (x-y)^2$ which cannot be negative $\therefore$ x^2+y^2 cannot be less than $2xy$.

Ex. 2. $(x+y)^2$ cannot be less than $4xy$.

For $(x+y)^2-4xy = x^2+2xy+y^2-4xy$
$= x^2-2xy+y^2$
$= (x-y)^2$ which cannot be negative $\therefore$ $(x+y)^2$ cannot be less than $4xy$.

Ex. 3. x^3+y cannot be less than x^2y+y^2x when x and y are positive.

For $(x^3+y^3)-(x^2y+y^2x) = (x^3-x^2y)+(y^3-y^2x)$
$= x^2(x-y)-y^2(x-y)$
$= (x^2-y^2)(x-y)$
$= (x-y)^2(x+y)$ which

cannot be negative, because $(x-y)^2$ is either zero or positive, and $x+y$ is positive.

$\therefore$ x^3+y^3 cannot be less than x^2y+y^2x.

Ex. 4. Which is greater $2x^4+1$ or $2x^3+x^2$ when x is positive.

$$\begin{aligned}(2x^4+1)-(2x^3+x^2) &= (2x^4-2x^3)-(x^2-1)\\ &= 2x^3(x-1)-(x+1)(x-1)\\ &= (2x^3-x-1)(x-1).\end{aligned}$$

Now, to determine whether these factors are of the same or of different signs, divide one by the other to discover the sign of the quotient, which gives $(2x^3-x-1)\div(x-1) = 2x^2+2x+1$ which is positive; $\therefore$ when x is any positive quantity except 1 the two factors have the same sign (Art. 23) and $2x^4+1 > 2x^3+x^2$.

ART. 38. The advantage of the method of treating inequalities used in Art. 37, depends upon the facility with which expressions can be resolved into their component factors. This in ordinary cases may be done—

I. By trial.

This is effected by separating the expression into partial products by combining terms which have a common factor, thus forming a new expression with fewer terms; then, if necessary, proceeding in like manner with the reduced expression, and so on until the ultimate product is obtained.

Ex. 1. Resolve $3a^3-3a^2b+ab^2-b^3$ into its factors.

Observing that $3a^2$ is a common factor of $3a^3$ and $3a^2b$, and b^2 of ab^2 and b^3, we have—

$$\begin{aligned}3a^3-3a^2b+ab^2-b^3 &= (3a^3-3a^2b)+(ab^2-b^3)\\ &= 3a^2(a-b)+b^2(a-b)\\ &= (3a^2+b^2)(a-b).\end{aligned}$$

Ex. 2.
$$\begin{aligned}2a^2-5a+3 &= (2a^2-2a)-(3a-3)\\ &= 2a(a-1)-3(a-1)\\ &= (2a-3)(a-1).\end{aligned}$$

Ex. 3.
$$\begin{aligned}4a^2-5ab+b^2 &= (4a^2-4ab)-(ab-b^2)\\ &= 4a(a-b)-b(a-b)\\ &= (4a-b)(a-b).\end{aligned}$$

II. By formulæ.

The formulæ most useful for this purpose, are—

(1) $x^2+(a+b)x+ab = (x+a)(x+b)$ for all values of x, a, and b.

(2) $x^2\pm 2ax+a^2 = (x\pm a)^2$.

(3) $x^2-a^2 = (x+a)(x-a)$.

Also, when n is any positive integer—

(4) $a^n-b^n = (a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\&c.)$

and, when n is odd—

(5) $a^n+b^n = (a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2-\dots$
$+a^2b^{n-3}-ab^{n-2}+b^{n-1})$

all which may be verified by actual multiplication. Formula (5) follows from (4) by writing $-b$ for b, which changes the sign of terms which contain odd powers of b; that is, b, b^3, $b^5 \dots . b^{n-2}$, b^n.

Also, by means of formulæ (2) and (3), we may resolve any expression of the form $x^2 \pm ax+b$ into its component factors, thus—

$$x^2 \pm ax+b = \left(x^2 \pm ax+\frac{a^2}{4}\right)-\left(\frac{a^2}{4}-b\right)$$
$$= \left(x \pm \frac{a}{2}\right)^2 - \left(\sqrt{\frac{a^2}{4}-b}\right)^2 \text{ by (2)}$$
$$= \left(x \pm \frac{a}{2}+\sqrt{\frac{a^2}{4}-b}\right)\left(x \pm \frac{a}{2}-\sqrt{\frac{a^2}{4}-b}\right) \text{ by (3).}$$

Example. Resolve $x^2+39x+374$ into its factors.

$$x^2+39x+374 = \left(x^2+39x+\frac{39^2}{4}\right)-\left(\frac{39^2}{4}-374\right)$$
$$= \left(x+\frac{39}{2}\right)^2-\left(\frac{5}{2}\right)^2$$
$$= (x+22)(x+17).$$

The following are examples of the application of the other formulæ. It will generally be most convenient first to separate those factors which are common to each term of the expression to be reduced; as, for instance, $a^3x^2-a^5$ into $a^3(x^2-a^2)$; and for even values of n to reduce a^n-b^n by (3) before applying (4).

Ex. 1. $x^2-9x+14 = x^2+(-7-2)x+(-7\times-2)$

$$= (x-7)(x-2) \text{ by } (1).$$

Ex. 2. $a^3bc+2a^2bc^2+abc^3 = abc(a^2+2ac+c^2)$

$$= abc(a+c)^2 \text{ by } (2).$$

Ex. 3. $ax^4y-a^5y^5 = ay(x^4-a^4y^4)$

$$= ay(x^2+a^2y^2)(x^2-a^2y^2)$$

$$= ay(x^2+a^2y^2)(x+ay)(x-ay) \text{ by } (3).$$

Ex. 4. $8x^3-27 = (2x)^3-3^3$

$$= (2x-3)(4x^2+6x+9) \text{ by } (4).$$

Ex. 5. $x^9+y^9 = (x^3)^3+(y^3)^3$

$$= (x^3+y^3)(x^6-x^3y^3+y^6)$$

$$= (x+y)(x^2-xy+y^2)(x^6-x^3y^3+y^6) \text{ by } (5).$$

Ex. 6. $x^6-64 = (x^3)^2-(2^3)^2 = (x^3+2^3)(x^3-2^3)$

$$= (x+2)(x^2-2x+4)(x-2)(x^2+2x+4)$$

by (3, 4, and 5).

39. The rule above given for resolving

x^2+ax+b into its factors will enable us readily to solve inequalities of the second degree.

Ex. 1. What values of x make $x^2+3x+\frac{3}{4}>2$.

Subtracting the less from the greater, we have—

$$\begin{aligned} x^2+3x-\tfrac{5}{4} &= (x^2+3x+\tfrac{9}{4})-1 \\ &= (x+\tfrac{3}{2})^2-1 \\ &= (x+\tfrac{5}{2})(x+\tfrac{1}{2}) \end{aligned}$$

which being positive, the factors $x+\frac{5}{2}$ and $x+\frac{1}{2}$ must have the same sign ; $\therefore$ x may have any value zero or positive, and any negative value numerically greater than $\frac{5}{2}$ or numerically less than $\frac{1}{2}$.

Ex. 2. What values of x make $x^2-8x>-25$.

Subtracting the less from the greater, we have—

$$\begin{aligned} x^2-8x+25 &= (x^2-8x+16)+9 \\ &= (x-4)^2+9 \text{ which is essentially positive.} \end{aligned}$$

$\therefore$ x may have any value zero, positive or negative.

Ex. 3. What values of x make $x^2-12x+46>10$.

Subtracting the less from the greater, we have—

$x^2-12x+36=(x-6)^2$, which is positive for all values of x except $x=6$; $\therefore$ x may have any values, zero positive or negative, except 6.

40. It is evident, from the preceding articles,

that all the definitions, processes, and relations of this algebra coincide, as far as positive values only are concerned, with the algebra of abstract numbers. Hence, an important use of this algebra is, that results purely arithmetical may logically be deduced without avoidance of impossible subtractions or the inconvenience of restricted rules. Thus, the equations of arithmetical algebra being regarded as equations between positive values, may be solved by intervention of negative symbols, or by any processes sanctioned by the present algebra, though arithmetically without meaning.

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